Between two and four values

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Introduction

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We shall investigate the lattice of super-Belnap logics.

Preliminaries: logics

A (finitary) logic is a relation between formulas and (finite) sets of formulas which satisfies the following:

$$\varphi \vdash \varphi \qquad \text{(identity)}$$

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Gamma, \Delta \vdash \varphi \qquad \text{(monotonicity)}$$

$$\text{if } \Delta \vdash \varphi \text{ and } \Gamma, \varphi \vdash \psi, \text{ then } \Gamma, \Delta \vdash \psi \qquad \text{(cut)}$$

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma\varphi \qquad \text{(structurality)}$$

We restrict to finitary logics throughout this presentation (almost).

Preliminaries: lattices of logics

If $\mathcal{L}_1 \subseteq \mathcal{L}_2$, we say that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$.

The extensions of a logic $\mathcal{L}$ form a lattice $\mathrm{Ext}\,\mathcal{L}$:

$\Gamma \vdash_{\mathcal{L}_1 \cap \mathcal{L}_2} \varphi$: $\Gamma \vdash_{\mathcal{L}_1} \varphi$ and $\Gamma \vdash_{\mathcal{L}_2} \varphi$
$\Gamma \vdash_{\mathcal{L}_1 \vee \mathcal{L}_2} \varphi$: φ provable from Γ using the rules of both logics

A logic is axiomatized by a set of rules (relative to $\mathcal{L}_0$) if it is the smallest logic (extending $\mathcal{L}_0$) which contains these rules.

Or in other words, if the logic coincides with what we can prove using substitution instances of these rules (and the rules of $\mathcal{L}_0$).

The Belnap–Dunn logic $\mathcal{B}$

In the logic $\mathcal{B}$, truth values are computed in a perfectly classical way:

$\varphi \wedge \psi$ is true $\Leftrightarrow$ φ is true and ψ is true

$\varphi \wedge \psi$ is false $\Leftrightarrow$ φ is false or ψ is false

$\varphi \vee \psi$ is true $\Leftrightarrow$ φ is true or ψ is true

$\varphi \vee \psi$ is false $\Leftrightarrow$ φ is false and ψ is false

$-\varphi$ is true $\Leftrightarrow$ φ is false

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$-\varphi$ is true $\Leftrightarrow$ φ is false

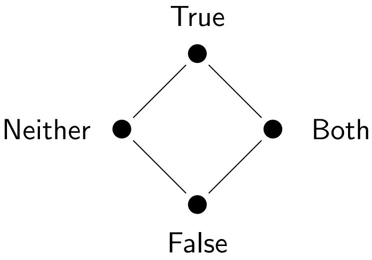
$-\varphi$ is false $\Leftrightarrow$ φ is true

. . . it's just that sentences may be both true and false or neither.

In other words, the truth and falsehood values are computed separately.

The Belnap–Dunn logic $\mathcal{B}$

These truth values may naturally be organized into a lattice as follows:



The lattice connectives are interpreted by the lattice structure. . .

. . . and negation rotates the lattice around the Neither–Both axis.

(Such structures are called de Morgan algebras.)

The semantics of super-Belnap logics

The consequence relation of $\mathcal{B}$ is defined classically:

$\Gamma \vdash_{\mathcal{B}} \varphi$ means: φ is true in each valuation in which all of Γ is true

Valuation means:	homomorphism $v : \mathbf{Fm} \to \mathbf{DM_4}$
True means:	truth value belongs to $\{True, Both\}$

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Let us generalize this a little:

A matrix $\mathbf{M}$ is an algebra $\mathbf{A}$ with a set of designated values $\mathcal{D} \subseteq \mathbf{A}$.

An $\mathbf{M}$-valuation is a homomorphism $v : \mathbf{Fm} \to \mathbf{A}$.

$\Gamma \vDash_{\mathbf{M}} \varphi$ means: $v[\Gamma] \subseteq \mathcal{D} \Rightarrow v(\varphi) \in \mathcal{D}$ for each $\mathbf{M}$-valuation v

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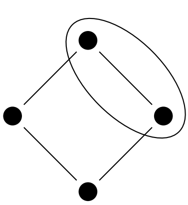
Each logic is given by a class of matrices K.

$\Gamma \vDash_{\mathsf{K}} \varphi$ means: $\Gamma \vDash_{\mathbf{M}} \varphi$ for each $\mathbf{M} \in \mathsf{K}$.

Our main question

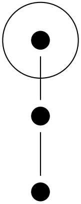
Which logics live between the Belnap–Dunn logic and classical logic?



$\mathcal{B}$ $\mathcal{CL}$

The three-valued cousins of $\mathcal{B}$

Dropping the truth value *Both* yields the following matrix:



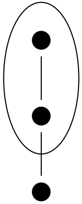
Observe: $p \vee q, -q \vee r \models p \vee r$

On the other hand: $\emptyset \not\models p \vee -p$

This yields Stephen C. Kleene's strong three-valued logic $\mathcal{K}$ (1938).

The three-valued cousins of $\mathcal{B}$

Dropping the truth value *Neither* yields the following matrix:



Observe: $\emptyset \models p \vee -p$

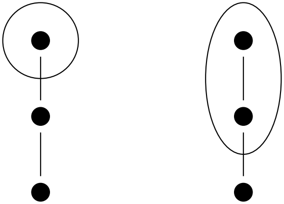
On the other hand: $p, -p \not\models q$

This yields Graham Priest's Logic of Paradox $\mathcal{LP}$ (1979).

The three-valued cousins of $\mathcal{B}$

Taking the intersection of $\mathcal{K}$ and $\mathcal{LP}$ yields Kleene's logic of order $\mathcal{K}_{\leq}$.

That is, $\Gamma \vdash_{\mathcal{K}_{\leq}} \varphi$ if and only if $\Gamma \vdash_{\mathcal{K}} \varphi$ and $\Gamma \vdash_{\mathcal{LP}} \varphi$.



Observe: $(p \wedge -p) \vee r \vDash q \vee -q \vee r$

On the other hand: $\emptyset \nvDash p \vee -p$ and $p, -p \nvDash q$

The Belnap–Dunn logic and normal forms

$\mathcal{B}$ may be viewed as the logic of normal forms (CNF and DNF).

Each formula has an essentially unique normal form in $\mathcal{B}$.

$\mathcal{LP}$ then allows for adding redundant disjunctions to CNFs. . .

. . . whereas $\mathcal{K}$ allows for removing redundant conjunctions from DNFs.

Super-Belnap logics may therefore be of some interest even to the classical logician: they allow us to study of the fine structure of classical logic.

The lattice of super-Belnap logics so far

$$
\begin{tikzcd}[column sep=small]
& \mathcal{TRIV} & \\
& \mathcal{CL} \arrow[u, dash] & \\
\mathcal{LP} \arrow[ur, dash] & & \mathcal{K} \arrow[ul, dash] \\
& \mathcal{K}_{\leq} \arrow[ul, dash] \arrow[ur, dash] & \\
& \mathcal{B} \arrow[u, dash] &
\end{tikzcd}
$$

$$\mathcal{K} = \mathcal{B} + \text{resolution}$$
$$\mathcal{LP} = \mathcal{B} + \text{excluded middle}$$
$$\mathcal{CL} = \mathcal{B} + \text{excluded middle} + \text{resolution}$$

Proof theory for super-Belnap logics

Let us have a look at sequent calculi for the above logics.

Interpretation: sequent $\Gamma \Rightarrow \Delta$ corresponds to formula $-\bigwedge \Gamma \vee \bigvee \Delta$.

The sequent calculus for the Belnap–Dunn logic:

- keeps the logical rules of classical logic
- includes their inverses, i.e. elimination rules
- keeps the structural rules of classical logic
- drops identity and cut

Identity corresponds to the axiom $p \vee -p$ (valid in $\mathcal{LP}$).

Cut corresponds to the resolution rule $p \vee q, -q \vee r \vdash p \vee r$ (valid in $\mathcal{K}$).

The following observations are due to A. Pynko (for $\mathcal{B}$ and $\mathcal{LP}$).

Sequent calculus for $\mathcal{B}$

Logical rules

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\varphi} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{-\varphi, \Gamma \Rightarrow \Delta}$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}$$

Structural rules

Exchange, Weakening, Contraction

Completeness theorems

Completeness theorems state:

sequent σ provable from a set of sequents $\Sigma \Leftrightarrow \tau(\sigma)$ follows from $\tau[\Sigma]$

Theorem

The above calculus is complete w.r.t. the logic $\mathcal{B}$.

For $\mathcal{K}$: add the cut rule
For $\mathcal{LP}$: add the identity axiom
For $\mathcal{CL}$: add both of the above

Given identity and cut, we can drop the elimination or introduction rules.

Hence why they are missing from the standard calculi for $\mathcal{CL}$...

...but they still show up in the proof of cut elimination (inversion lemma)!

Cut elimination: non-classical proof

Cut elimination theorems state:

the cut rule is redundant when proving a sequent from $\emptyset$

Non-classical proof of cut elimination:

(1) elimination rules are redundant even without cut (inversion lemma)
(2) $\mathcal{LP}$ and $\mathcal{CL}$ have the same theorems (easy semantic argument)
(3) calculus for $\mathcal{LP}$ minus elimination rules $=$ calculus for $\mathcal{CL}$ minus cut

Not only do we get a non-classical proof of cut elimination, but this reasoning immediately suggests the following dualization...

Identity elimination: non-classical proof

Identity elimination theorems state:

the identity axiom is redundant when proving $\emptyset$ from a set of sequents

More precisely, to prove a set of sequents to be classically inconsistent:

- we do not need the identity axiom and the introduction rules
- we do need cut and the elimination rules

Non-classical proof of identity elimination:

(1) introduction rules are redundant even without identity

(2) $\mathcal{K}$ and $\mathcal{CL}$ have the same antitheorems

(3) $\mathcal{K}$ minus introduction rules $= \mathcal{CL}$ minus identity

(here $\mathcal{CL} =$ the calculus with elimination instead of introduction rules)

Other super-Belnap logics

Back to our programme of investigating the lattice of super-Belnap logics.

The logics introduced above have all been around at least since the 1970s.

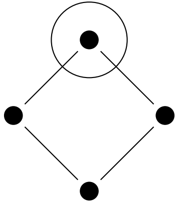
They are used e.g. in gappy (Kripke) and glutty (Priest) theories of truth.

Until recently, these were essentially the only known super-Belnap logics.

Pietz & Rivieccio (2013):	Exactly True Logic $\mathcal{ETL}$
Rivieccio (2012):	an infinity of super-Belnap logics
Our result:	a continuum of super-Belnap logics...
	...described in graph-theoretic terms

One more logic: $\mathcal{ETL}$

Changing the set of designated values of $\mathcal{B}$ yields the matrix:



This logic extends $\mathcal{B}$ by $p, -p \vee q \vdash q$ (the disjunctive syllogism).

This is Pietz and Rivieccio's Exactly True Logic $\mathcal{ETL}$ (2013).

Miscellaneous results on Ext $\mathcal{B}$

$\mathcal{CL} = \mathcal{LP} \vee \mathcal{ETL}$ canonically:

if $\mathcal{CL} = \mathcal{L}_1 \vee \mathcal{L}_2$, then $\mathcal{L}_1 \supseteq \mathcal{LP}$ and $\mathcal{L}_2 \supseteq \mathcal{ETL}$ or vice versa

$\mathcal{B}$ has a smallest extension: $p, -p \vdash q \vee -q$

$\mathcal{K}$ has a largest sublogic in Ext $\mathcal{ETL}$: $\chi_n \vee q, -q \vee r \vdash r \quad (n \in \omega)$

$$\chi_n = (p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge -p_n)$$

Each logic is either below $\mathcal{K}$ or above $\mathcal{LP}$.

The only structurally complete super-Belnap logics are $\mathcal{K}$ and $\mathcal{CL}$.

Explosive extensions

Recall: $\chi_n = (p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge -p_n)$

Define:

$$\mathcal{ECQ}_n = \mathcal{B} + \chi_n \vdash \emptyset \qquad \mathcal{ETL}_n = \mathcal{ETL} \vee \mathcal{ECQ}_n$$

$$\mathcal{ECQ}_\omega = \bigcup_{n \in \omega} \mathcal{ECQ}_n \qquad \mathcal{ETL}_\omega = \mathcal{ETL} \vee \mathcal{ECQ}_\omega$$

Explosive extensions are extension by rules of the form $\Gamma \vdash \emptyset$.

Explosive extensions of $\mathcal{L}$ form a distributive lattice $\operatorname{Exp}\operatorname{Ext}\mathcal{L}$.

Explosive super-Belnap logics are explosive extensions of $\mathcal{B}$.

Graph theory meets super-Belnap logics

By graphs we mean undirected graphs, possibly with loops.

Theorem

$\mathrm{Exp}\,\mathrm{Ext}\,\mathcal{ETL}$ ($\mathrm{Exp}\,\mathrm{Ext}\,\mathcal{B}$) *is dually isomorphic to the lattice of classes of finite graphs closed under homomorphisms (with an extra top element).*

Theorem

The interval $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ *is dually isomorphic to the lattice of classes of finite graphs closed under surjective homomorphisms, finite disjoint unions, and vertex reductions (contracting all outgoing edges of a vertex).*

In fact, the whole lattice $\mathrm{Ext}\,\mathcal{B}$ may be described in analogical terms.

Some corollaries

Corollary

There is a continuum of (explosive) super-Belnap logics.
$\Rightarrow$ *There is a continuum of antivarieties of de Morgan algebras.*
There are infinitary (explosive) super-Belnap logics.
$\Rightarrow$ *There are infinitary antivarieties of de Morgan algebras.*

Proof.

Uses the countable universality of the hom order on graphs. $\square$

Corollary

The logics $\mathcal{ETL}_{n+1}$ *are not complete w.r.t. any finite set of finite matrices.*

Proof.

Uses the classical theorem of Erdős (1959) that there are graphs of arbitrarily large girth among non-n-colourable graphs. $\square$

Conclusion

Super-Belnap logics are a largely unexplored area of non-classical logic. . .

. . . which may provide insight into the fine structure of classical logic.

Surprising connections between super-Belnap logics and finite graphs. . .

. . . graph-theoretic methods may be used to study super-Belnap logics.

In particular, the lattice of explosive super-Belnap logics is essentially the dual of the lattice of classes of finite graphs closed under homomorphisms.

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Thank you for your attention.